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Observable consequences of gauge invariant electromagnetic-torsion coupling — a departure from Einsteinian gravity

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Abstract : Presence of a gauge invariant coupling of electromagnetic field with spacetime torsion in a string inspired effective field theory is discussed. Departure from the estimates of classical tests of General Relativity is predicted because of such a coupling. It is further shown that the observed optical rotation of the plane of polarization of distant galactic radiation in excess of the Faraday rotation can be explained in such a scenario

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1. Introduction

Gravitational theories in a curved background space-time with torsion has been an area of investigation for a long time. Torsion, appears as an antisymmetric tensor part in the space-time connection [1] and is an inescapable consequence when the matter fields giving rise to space-time curvature are possessed with spin [2,3]. Therefore, a theory with torsion provides an effective classical background for quantum matter fields. The resulting theory, known as Einstein-Cartan theory, is not gauge invariant if one introduces electromagnetic coupling in a space-time endowed with asymmetric connection through the usual minimal coupling scheme. A possible resolution to this problem was found in the context of a string theory inspired torsion space-time where torsion appears in the form of a third rank antisymmetric tensor field strength corresponding to the second rank antisymmetric massless mode of string called Kalb-Ramond (KR) field [4,6]. Such a resolution is crucial to study the observable effects of torsion through experiments by electromagnetic fields in different scenarios.

In this review we first discuss how torsion can be coupled to electromagnetism in a gauge invariant way. Subsequently we find the most general spherically symmetric solution and re-examine the well known classical test of General Relativity in such a modified bakground. We then offer a possible explanation for the unexplained optical rotation of the plane of polarization of the distant galactic radio waves.

Thus, torsion is, in some sense, an inherent feature in the low-energy effective string action.

2. Gauge invariant electromagnetic coupling with torsion

Extensive studies have already been carried out regarding the coupling of torsion with other spin fields, especially the electromagnetic field, where the well-known problem of violation of $U(1)$ gauge-invariance [7] is explored. We now describe the results obtained in [4] where the existence of a gauge invariant coupling between electromagnetic field and the KR field was shown in the context of a string inspired model.

The electromagnetic field strength is defined by

$$F_{\mu\nu} = D_{\mu}A_{\nu} - D_{\nu}A_{\mu} \tag{1}$$

where the covariant derivative is given as

$$D_{\mu}A_{\nu} = \partial_{\mu}A_{\nu} + \Gamma^{\rho}_{\mu\nu}A_{\rho}. \tag{2}$$

Here the affine connection Γ includes torsion. The expression for $F_{\mu\nu}$ in this case turns out to be

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} - 2T^{\rho}_{\mu\nu}A_{\rho} \tag{3}$$

This expression is not invariant under the standard $U(1)$ electromagnetic gauge transformation $\delta A_\mu = \partial_\mu \omega$, which is naturally unacceptable as a consistent field theory of electromagnetism and torsion This forces one to conclude that electromagnetic field has no coupling to the antisymmetric part of the affine connection

The situation changes completely in the framework of the low energy effective field theory of string inspired model where there exists yet another non-gravitational field, possibly massless, to function as the source of the torsion Within the option of bosonic fields, the Kalb-Ramond (KR) antisymmetric second rank tensor field $B_{\mu\nu}$ appears as a possible candidate $B_{\mu\nu}$, is a massless second rank antisymmetric field with a third rank antisymmetric tensor field strength and has the following gauge transformation $\delta B_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ which leaves its field strength $H_{\mu\nu\lambda} \equiv \partial_{[\mu}B_{\nu\lambda]}$ gauge invariant The gauge invariance mentioned above is of course well-known in the string context To obtain a coupling that is invariant under both electromagnetic and Kalb-Ramond gauge transformations, the KR tensor potential must be endowed with a non-trivial electromagnetic gauge transformation property, and the KR field strength must accordingly be modified with the addition of an electromagnetic Chern Simons three tensor. Remarkably such a term appears naturally from the requirement of quantum consistency of the underlying string theory [6], where the KR 3-form $H_{\mu\nu\lambda}$ is modified by addition of Yang-Mills and gravitational Chern Simons 3-forms to ensure that the quantum theory is anomaly free Thus, the modified KR field strength three-tensor in our case is defined as

$$\tilde{H}_{\mu\nu\lambda} \equiv H_{\mu\nu\lambda} + \frac{1}{3}A_{[\mu}F_{\nu\lambda]} \tag{4}$$

This modified tensor $\tilde{H}_{\mu\nu\lambda}$ is gauge invariant under standard $U(1)$ gauge transformations, provided we stipulate that the KR potential transforms under $U(1)$ gauge transformations as $\delta B_{\mu\nu} = -\omega F_{\mu\nu}$. We have of course used the standard Bianchi identities for the Maxwell field involving the Christoffel connection. Further, the Maxwell field is assumed to be invariant under KR gauge transformations defined earlier

We now propose the following action for a manifestly gauge invariant Einstein-Cartan-Maxwell-Kalb-Ramond coupling,

$$S = \int d^4x\sqrt{-g}\left[R(g,T) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\tilde{H}_{\mu\nu\lambda}\tilde{H}^{\mu\nu\lambda} + T^{\mu\nu\lambda}\tilde{H}_{\mu\nu\lambda}\right]$$

where R is the scalar curvature, defined as R $R_{\alpha\mu\beta\nu}g^{\alpha\beta}g^{\mu\nu}$ and $R_{\alpha\mu\beta\nu}$ is the Riemann-Christoffel tensor -

$$R^{\kappa}_{\mu\nu\lambda} = \partial_\mu \Gamma^{\kappa}_{\nu\lambda} - \partial_\nu \Gamma^{\kappa}_{\mu\lambda} + \Gamma^{\kappa}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\kappa}_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda}$$

The torsion tensor $T_{\mu\nu\lambda}$ is an auxiliary field in eq obeying the constraint equation

$$T_{\mu\nu\lambda} = \tilde{H}_{\mu\nu\lambda} \tag{7}$$

Thus, the augmented KR field strength three tensor plays the role of the spin angular momentum density [1] Substituting the above equation in (5) and varying with respect to $B_{\mu\nu}$ and A_μ respectively, we obtain the equations

$$D^{S}_{\mu}\tilde{H}^{\mu\nu\lambda} = 0 \tag{8}$$

and

$$D^{S}_{\mu}F^{\mu\nu} = \tilde{H}^{\mu\nu\lambda}F_{\lambda\mu}, \tag{9}$$

where, D^S is the covariant derivative using the Christoffel connection Clearly, these equations of motion are manifestly gauge covariant under both gauge transformations The interaction term thus has the structure

$$S_{int} = \int d^3x\sqrt{-g}\,H^{\mu\nu\lambda}A_\mu F_{\nu\lambda} \tag{10}$$

A similar structure has been proposed earlier [8] on quite different grounds to solve the problem of gauge invariant Einstein-Cartan-Maxwell couplings Since the KR three tensor is Hodge-dual to the derivative of a spinless field H, so that, after a partial integration, one obtains,

$$S_{int} = \frac{1}{2}\int d^4x H F_{\mu\nu}\,{}^*F^{\mu\nu}, \tag{11}$$

where, ${}^*F^{\mu\nu} \equiv \in^{\mu\nu\lambda\sigma} F_{\lambda\sigma}$ Here, we have noted the fact that

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,{}^*F^{\mu\nu}\right) = D^S_\mu\,{}^*F^{\mu\nu} = 0 \tag{12}$$

by the Maxwell Bianchi identity It may be noted that the gauge Chern Simon's term is the crucial ingredient in inducing gauge invariant couplings of the electromagnetic field

3. Rotation of the plane of polarization

We now examine the consequences such a coupling, to astrophysical/cosmological observables. The present section reviews the work presented in [7] where it has been explicitly shown that the KR field may induce a rotation of the plane of polarization of electromagnetic radiation from cosmologically distant sources. There is some evidence that optical activity of a related type may have already been ***observed*** in radiation from distant quasars and other radio sources [9–11]. The observed angle of rotation of the plane of polarization can be expressed as

$$\theta = \alpha\lambda^2 + \chi \tag{13}$$

where α (the Faraday rotation measure) and χ are constants and λ is the wavelength of the electromagnetic wave. χ is the angle between a reference axis and the electric field of the wave when it is emitted from the source galaxy, while the first term, because of its quadratic dependence on the wavelength represents ***Faraday*** rotation of the plane of polarization due to passage of the electromagnetic wave through galactic (and possibly inter-galactic) magnetized plasmas. Here we show that Einstein-Kalb-Ramond-Maxwell coupling can be responsible, in explaining the origin of this extra bit of rotation (χ). Since the KR field appears naturally in some supergravity theories and hence in the massless spectrum of closed string theory, therefore such an observation may perhaps be considered as an evidence for supergravity as well as being a hint of an underlying string structure.

In this analysis we treat the KR field as a *tiny* perturbation on the Maxwell field equations in a standard cosmological background such that it's back-reaction on the background space-time geometry can be neglected. We choose two standard scenarios, *viz*, the *spatially* flat Friedmann-Robertson-Walker background with the scale factor, which depends only on (co-moving) time, evolving according to both a radiation dominated and a matter dominated scheme. In general, in addition to the graviton and the KR field, the perturbative sector of the heterotic string contains a scalar dilaton field whose dynamics is also known to have cosmological consequences. Here we shall however ignore this dynamics for the moment by freezing the dilaton and focus instead on the effects of the KR field alone on the electromagnetic wave. Taking the KR field strength to be

$$H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}{}^{\sigma} D_{\sigma} H,$$

where H is a pseudo-scalar, one obtains the modified generally covariant Maxwell equations [4]

$$\begin{aligned} &\boldsymbol{D}\cdot\boldsymbol{E} = 2\sqrt{G}\boldsymbol{D}H\cdot\boldsymbol{B} \\ &D_0\boldsymbol{E} - \boldsymbol{D}\times\boldsymbol{B} = -2\sqrt{G}[D_0H\boldsymbol{B} - \boldsymbol{D}H\times\boldsymbol{E}] + 2G[(\boldsymbol{B}^2 - \boldsymbol{E}^2)\boldsymbol{A} \\ &\qquad +(\boldsymbol{A}\cdot\boldsymbol{E})\boldsymbol{E} + (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{B}] \\ &D_0\boldsymbol{B} + \boldsymbol{D}\times\boldsymbol{E} = 0 = \boldsymbol{D}\cdot\boldsymbol{B} \end{aligned} \tag{14}$$

Here D_μ is the covariant derivative in the spatially flat FRW metric. To a first approximation, we ignore the $O(G)$ terms which arise from the Chern-Simons augmentation of the KR field strength. We also absorb the $\sqrt{G}$ in the pseudo-scalar field H to render this field dimensionless. The equations now become,

$$\begin{aligned} &\boldsymbol{D}\cdot\boldsymbol{E} = 2\boldsymbol{D}H\cdot\boldsymbol{B} \\ &D_0\boldsymbol{E} - \boldsymbol{D}\times\boldsymbol{B} = -2[D_0H\boldsymbol{B} - \boldsymbol{D}H\times\boldsymbol{E}] \\ &D_0\boldsymbol{B} + \boldsymbol{D}\times\boldsymbol{E} = 0 = \boldsymbol{D}\cdot\boldsymbol{B} \end{aligned} \tag{15}$$

The last equations in the array are the Maxwell Bianchi identity. In a spatially flat isotropic FRW background with metric

$$ds^2 = R^2(\eta)(d\eta^2 - dx^2), \tag{16}$$

where, η is the conformal time coordinate, defined by $d\eta = dt/R(t)$, the above equations assume the form,

$$\begin{aligned} &\nabla\cdot\boldsymbol{E}R^2 = 2\nabla H\cdot\boldsymbol{B}R^2 \\ &\partial_\eta(\boldsymbol{E}R^2) - \nabla\times\boldsymbol{B}R^2 = -2[\partial_\eta H\boldsymbol{B}R^2 - \nabla H\times\boldsymbol{E}R^2] \\ &\partial_\eta(\boldsymbol{B}R^2) - \nabla\times\boldsymbol{E}R^2 = 0 = \nabla\cdot\boldsymbol{B}R^2 \end{aligned} \tag{17}$$

We first consider a *flat* background space-time ($R(\eta) = 1$), to obtain a preliminary understanding of the effects involved. It may be recalled that the KR field strength $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$, so that it satisfies the Bianchi identity

$$\epsilon^{\mu\nu\lambda\sigma}\,\partial_\sigma H_{\mu\nu\lambda} = 0 \tag{18}$$

This implies that the pseudo-scalar $H = 0$ satisfies the massless Klein-Gordon eqn $\Box H = 0$. For non-flat backgrounds, the d'Alembertian operator is to be replaced by its generally covariant counterpart.

Assuming that H is only a function of the co-moving time coordinate η, the Klein-Gordon equation reduces to the simple equation $d^2H/d\eta^2 = 0$ with the obvious solution $H = h\eta + h_0$, where h and h_0 are constants. This spatial homogeneity of the Klein-Gordon field is possibly a justified assumption over the cosmologically long distance scales of our interest.

Following the approaches adopted in [12] and [13], we arrive at the equation

$$\frac{d^2b_\pm}{d\eta^2} + (k^2 \mp 2hk)b_\pm = 0, \tag{19}$$

where we have decomposed $B = b(\eta)e^{ik\cdot x}$ and have chosen the z direction to be the propagation direction of the electromagnetic wave The circular polarization states are defined as $b_\pm \equiv b_x \pm ib_y$ Unlike the corresponding equation in [12], eq (19) can be solved *exactly*

$$b_\pm = b_0 e^{i\omega_\pm\eta} \equiv b_0 e^{i\phi_\pm}, \tag{20}$$

where, $\omega_\pm^2 \equiv k(k \mp 2h)$ The optical activity due to the presence of the KR field is thus given by the difference

$$(\Delta\phi)_{\rm mag} \equiv \frac{1}{2}(\phi_+ - \phi_-) = -h\eta \text{ for } k \gg h \tag{21}$$

The equation for the electric field (with an assumption $E = e(\eta)e^{ikz}$ and a similar definition for $e_\pm$) takes the form

$$\frac{d^2 e_\pm}{d\eta^2} + k^2 e_\pm = -2h\frac{db_\pm}{d\eta} \tag{22}$$

This implies that a solution of the electric field equation is therefore dependent on the solution of the magnetic field equation It is easy to see that,

$$e_\pm = \mp\frac{b_0}{k}\omega_\pm i e^{i\omega_\pm\eta} \tag{23}$$

is a solution for the electric equation and the amount of rotation is the same for both the electric and magnetic fields

Our result clearly indicates that KR field indeed induces optical activity

We now generalize this to spatially flat FRW universe in both radiation and matter dominated cases So we need to solve Maxwell equations in a non-trivial cosmology where we choose for simplicity the spatially flat Friedman-Robertson-Walker (FRW) type of background

The equation of motion of the pseudo-scalar field once again is given as

$$\Box H = 0. \tag{24}$$

where $\Box$ is now the covariant d'Alembertian corresponding to the spatially flat metric

Once again considering spatially homogeneous H field, such that $H = H(\eta)$ we arrive at,

$$\partial_0 H = -\frac{h}{R^2(\eta)} \tag{25}$$

where h is an integration constant, which, in a sense, is a 'measure' of the strength of the pseudoscalar H field or, equivalently, the dual three form field $H_{\mu\nu\lambda}$.

The equations that the polarization states $b_\pm$ satisfy for such a background can be similarly written down in terms of the quantity $F_\pm$ where $b_\pm = F_\pm/R^2$ They are

$$\frac{d^2 F_\pm}{d\eta^2} + \left(k^2 \mp \frac{2hk}{R^2(\eta)}\right)F_\pm = 0 \tag{26}$$

Similar to the flat space-time case, a corresponding equation for the electric field polarization states $e_\pm$ can also be obtained in terms of a quantity $G_\pm$ where $e_\pm = G_\pm/R^2$ This turns out to be

$$\left(\frac{d^2}{d\eta^2} + k^2\right)G_\pm = -2h\frac{d}{d\eta}\left[\frac{F_\pm}{R^2}\right]$$

It may be noted that the electric field equations are dependent on the solution of their magnetic field counterparts. The rotation can be calculated for both the electric and magnetic fields and we show our results only for the magnetic field case

The equations can be solved explicitly once the scale factor $R(\eta)$ is known For a radiation dominated FRW model we assume a scale factor $R(\eta) = \eta/\eta_0^R$ for $1/\eta_0^R = (8\pi G\epsilon_0/3)^{1/2}$, with ϵ_0 being the primordial radiant energy density Similarly for a matter dominated model we assume $R(\eta) = (\eta/\eta_0^M)^2$ Our objective is to find the asymptotic dependence on η and the parameters of the theory which are $\bar{h} = h(\eta_0^R)^2, h' = h(\eta_0^M)^4$ and the wave number k

Accordingly, we have the two equations for the radiation and matter dominated cases as given below

$$\frac{d^2 F_\pm}{dx^2} + \left(1 - \frac{\mu_\pm^2}{x^2}\right)F_\pm = 0,\ \mu_\pm^2 \equiv 2\bar{h}k \tag{28}$$

and

$$\frac{d^2 F_\pm}{dx^2} + \left(1 - \frac{\mu_\pm^2}{x^4}\right)F_\pm = 0,\ \mu_\pm^2 \equiv 2h'k^3 \tag{29}$$

In the above, we use dimensionless quantities throughout, with $x = k\eta$.

Using the ansatz

$$F_\pm(x) = e^{ix}\upsilon_\pm(x) \tag{30}$$

so that, eqs (28) and (29) reduce to

$$\frac{d^2\upsilon_\pm}{dx^2} + 2i\frac{d\upsilon_\pm}{dx} - \frac{\upsilon_\pm^2}{x^2}\upsilon_\pm = 0, \tag{31}$$

$$\frac{d^2\upsilon_\pm}{dx^2} + 2i\frac{d\upsilon_\pm}{dx} - \frac{\upsilon_\pm^2}{x^4}\upsilon_\pm = 0 \tag{32}$$

As we are only interested in asymptotic solution of these equations for $x \to +\infty$ therefore we choose a solution for both cases of the type

$$\upsilon_{+}(x) = \upsilon_0^{\pm} + \frac{\upsilon_1^{\pm}}{x} + \frac{\upsilon_2^{\pm}}{x^2} + \cdots \tag{33}$$

We use this ansatz is used to calculate the angle of rotation of the polarization plane, to lowest non-trivial order in $1/x$ (where $x = k\eta$). The result turns out to be,

$$\Delta\phi = \left|\arg\upsilon_0' - \arg\upsilon_0 + 2\tan^{-1}(\bar{h}k/x)\right| \text{ for } RD \tag{34}$$

and

$$\Delta\phi = \left|\arg\upsilon_0' - \arg\upsilon_0 + 2\tan^{-1}(h'k^3/3x^3)\right| \text{ for } MD \tag{35}$$

Recalling that the angle of rotation must be zero in absence of the the interaction, we get $\arg\upsilon_0' - \arg\upsilon_0 = 0$ This yields the angles of rotation as

$$\Delta\phi = \left|2\tan^{-1}(\bar{h}/3\eta)\right| \text{ for } RD \tag{36}$$

and

$$\Delta\phi = \left|2\tan^{-1}(h'/3\eta^3)\right| \text{ for } MD \tag{37}$$

For very small h, the inverse tangent may be replaced by its argument Our predictions from theory (to the lowest order in h) for the rotation angle thus may be checked against the data Rewriting the expressions in terms of the red-shift z, the expression for the look-back time can be obtained from the expression,

$$t - t_0 = \frac{2}{3H_0}\left[1-(1+z)^{-3/2}\right], \tag{38}$$

where H_0 is the value of the present Hubble parameter The relation between conformal time η and real time t can be obtained from the expression, $a(\eta)d\eta = dt$

One can similarly derive the expressions for the rotation of the electric field which to the lowest order turns out to be the same as for the magnetic field We further note the fact that at the lowest order $\boldsymbol{E}.\boldsymbol{B}$ is equal to zero, but it may not be so beyond this order It is interesting to note that KR field modifies many of the well-known properties of electromagnetic wave in vacuum as the presence of the KR field effectively makes the space optically active. We have thus arrived at the following situation A part of the rotation of the plane of polarization of light emitted from distant galaxies obtained after subtracting out the Faraday component can be explained from the coupling of the electromagnetic wave with the background Kalb-Ramond field We have explicitly written the modified Maxwell equations after the inclusion of the effects of the H field and analyzed both in flat and curved FRW backgrounds The wave equations for the electric and magnetic fields are indeed different (unlike usual electromagnetism) and the effects on the rotation are generally different

All this is perhaps a vindication of the proposal of supergravity and what one is observing is perhaps a massless mode of an underlying string theory, hitherto unobserved because of its weak coupling to other matter As pointed out in [14], such a weak coupling could be detected perhaps in future, if not through presently available data We leave open the question whether the present data does actually substantiate our theoretical conclusions Apart from the astrophysical ramifications of our work, the fact that the only known proposal of coupling the Maxwell field to an Einstein-Cartan geometry in a gauge invariant manner leads directly to the optical activity discussed above can, in principle, be of significant use in the detection of torsion as a geometrical property of space-time

We now carry out the most general study of the existence of possible static spherical symmetric asymptotically flat solutions of the vacuum field equations in presence of KR field Our aim is to establish a physically meaningful general solution which matches the limiting requirements and provide a proper understanding about how the standard Schwarzschild solution gets modified in the presence of different forms of the torsion field [18] Implications thereof, are obtained through the study of geodesic motion in such space-times We also investigate the corrections inflicted by torsion on some of the standard tests of general relativity theory [19] We repeat that although the full low energy effective action of string theory includes the graviton, dilaton, axion as well as other fields which may arise out of different type of compactifications, here we attempt to focus on the effects of the axion only We investigate whether the presence of axion (appearing as a dual field to the Kalb-Ramond induced torsion) is perceptible at all through the various observational and solar system tests

4. Static spherically symmetric solutions in a Kalb-Ramond background

In this section we review the work reported in [19] Following the identification [4] of the totally antisymmetric torsion tensor with the modified KR field strength $H_{\mu\nu\lambda}$, the action for gauge-invariant EC-KR coupling is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R(g)}{\kappa} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}\right] \tag{39}$$

This action resembles to the action for the low-energy effective string theory $R(g)$ is the Ricci scalar curvature and $\kappa \sim (Planck\ mass)^{-2}$ is the gravitational coupling constant. The modified KR field strength three-form H is defined by the KR field strength augmented by the $U(1)$ electromagnetic CS three-form $H = dB + \sqrt{\kappa} A \wedge F$. Due to the Planck mass suppression we neglect the CS term in the present analysis. The field equations that can be obtained from the above action are given as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}, \tag{40}$$

$$D_\mu H^{\mu\nu\lambda} \equiv \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} H^{\mu\nu\lambda} \right) = 0, \tag{41}$$

where $R_{\mu\nu}$ is the Ricci tensor of Riemannian geometry, and $T_{\mu\nu}$ is a symmetric 2-tensor, analogous to the energy-momentum tensor, and is given by

$$T_{\mu\nu} = \frac{1}{4}\left(3 g_{\nu\rho} H_{\alpha\beta\mu} H^{\alpha\beta\rho} - \frac{1}{2} g_{\mu\nu} H_{\alpha\beta\gamma} H^{\alpha\beta\gamma} \right) \tag{42}$$

We take the line element in its most general spherically symmetric form

$$ds^2 = e^{\nu(r,t)} dt^2 - e^{\lambda(r,t)} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2) \tag{43}$$

and expressing the three-form $H_{\mu\nu\lambda}$ in terms of its Hodge-dual one-form – a pseudo-vector – with independent components H_{012}, H_{013}, H_{023} and H_{123}, it has been shown in [18] that static spherical symmetric solutions, consistent with the basic requirement of asymptotic flatness, are possible only when $H_{023} \neq 0$ and all other components vanish. In this situation we have defined the dual pseudo-scalar 'axion' H as,

$$H_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda}{}^{\sigma} \partial_\sigma H, \tag{44}$$

where H depends on the radial coordinate r only. Denoting $H_{023}H^{023}$ by $[h(r)]^2$ the field equations can be expressed as

$$e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = \bar{\kappa} h^2, \tag{45}$$

$$e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r} \right) - \frac{1}{r^2} = -\bar{\kappa} h^2, \tag{46}$$

$$e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} + \frac{\nu' - \lambda'}{r} \right) = -2\bar{\kappa} h^2, \tag{47}$$

$$\partial_1\left(r^2 e^{\nu - \lambda/2} H' \right) = \partial_1 \left(r^2 h e^{\nu/2} \right) = 0, \tag{48}$$

where a prime indicates derivative with respect to r, and the constant $\bar{\kappa} = \frac{3}{4}\kappa$. The last equation in the above set is obtained by using both eq (3) and the Bianchi identity for the KR field, *viz*, $\epsilon^{\mu\nu\lambda\sigma} \partial_\sigma H_{\mu\nu\lambda} = 0$. The above equations can now be solved to obtain

$$h(r) = H'(r) e^{-\lambda/2} = \frac{b_0}{r^2} e^{-\nu/2} \tag{49}$$

and

$$e^{-\lambda} = 1 + \frac{c_1}{r} + \frac{\tau(r)}{r}, \tag{50}$$

$$e^{\nu} = \frac{c_2}{r(r + \tau(r) + c_1)} \exp\left[\int^r \frac{2dr}{r + \tau(r) + c_1} \right], \tag{51}$$

where b_0, c_1 and c_2 are the constants of the integrations and

$$\tau(r) = \bar{\kappa} \int^r r^2 h^2(r) dr \tag{52}$$

The above solutions are consistent only when they satisfy the asymptotic flatness requirement, *viz*, $e^{\pm\nu}$, $e^{\pm\lambda} \to 1$ as $r \to \infty$, and a consistency condition derived from the field equations

$$\tau'' + \frac{\tau'}{r} = \frac{\tau'(\tau' - 1)}{r + c_1 + \tau} \tag{53}$$

The asymptotic flatness condition on the solutions requires $c_2 = 1$. This can readily be verified in the limit where torsion vanishes, *i.e.*, $\tau(r) = 0$. For non-zero torsion, if we further put $c_1 = 0$, then as is shown in [18], a typical exact solution satisfying the above requirement can be obtained for a specific form of $\tau(r)$, *viz*, $\tau(r) = -b/r$, $b = \bar{\kappa} b_0^2 =$ constant (*i.e.*, $h(r) \sim 1/r^2$), whence

$$e^{-\lambda} = 1 - \frac{b}{r^2}, \tag{54}$$

$$e^{\nu} = 1, \tag{55}$$

and we have a wormhole for a real KR field, *i.e.*, a positive b. Note that this geometry has been discussed many times in the literature beginning with the work of Ellis [22] though its appearance in the context of the Kalb-Ramond field coupled to gravity had not been noticed till recently.

To enquire the uniqueness of our solution we take a general functional form of $\tau(r)$, which depends on the KR field strength $h(r)$. We also take the KR field to be real. As $\tau(r)$ does not involve any additive constant, we can express it in the form

$$\tau(r) = \sum_{m=1}^{\infty} a_m r^m + \sum_{n=1}^{\infty} \frac{b_n}{r^n} \quad (56)$$

Also using previously obtained equations we find

$$\tau'(r) = \frac{b}{r^2} e^{-\nu}, \; b = \bar{\kappa} b_0^2 \quad (57)$$

Since $e^{-\nu} \equiv g^{00} \to 1$ as $r \to \infty$, therefore from consistency requirements we find that all the a_m's in $\tau(r)$ vanish, i.e., $\tau(r) = \sum_{n=1}^{\infty} b_n / r^n$. Plugging this in eq. (15) and matching the coefficients of equal powers of r from both sides, we obtain

$$\tau(r) = b_1 \left[\frac{1}{r} - \frac{c_1}{2r^2} + \frac{c_1^2}{3r^3} - \left(1 - \frac{b_1}{6c_1^2}\right)\frac{c_1^3}{4r^4} + \left(1 - \frac{b_1}{2c_1^2}\right)\frac{c_1^4}{5r^5} - \left(1 - \frac{59b_1}{60c_1^2} + \frac{3b_1^2}{80c_1^4}\right)\frac{c_1^5}{6r^6} + \cdots \right] \quad (58)$$

Computing $\tau'(r)$ we find $b_1 = -b$ and the solutions can be expressed as

$$g_{00}(r) = 1 + \frac{c_1}{r} + \frac{bc_1}{6r^3} - \frac{bc_1^2}{6r^4} + \frac{bc_1^3 + 3b^2c_1}{40r^5} + \cdots \quad (59)$$

$$g^{11}(r) = 1 + \frac{c_1}{r} - \frac{b}{r^2} + \frac{bc_1}{2r^3} - \frac{bc_1^2}{3r^4} + \left(bc_1^3 + \frac{b^2c_1}{6}\right)\frac{1}{4r^5} + \cdots \quad (60)$$

while the solution for the KR field is given by

$$h(r) = \sqrt{\frac{b}{\bar{\kappa}}} \frac{1}{r^2} \left[1 - \frac{c_1}{r} + \frac{c_1^2}{r^2} - \left(c_1^3 + \frac{bc_1}{6}\right)\frac{1}{r^3} + \left(c_1^4 + \frac{bc_1^2}{2}\right)\frac{1}{r^4} + \cdots \right] \quad (61)$$

The above solutions are, by construction, asymptotically flat and reproduce the exact solution found in [18] for $c_1 = 0$. For $c_1 \neq 0$, we obtain the standard Schwarzschild solution, *viz.*, $e^{\nu} = e^{-\lambda} = 1 - r_s/r$ in the zero-torsion limit, i.e. $b = 0$, provided $-c_1 = r_s = 2GM$ (the Schwarzschild radius). In fact, whenever c_1 is non-vanishing, being a constant we can always identify it with $-r_s$, thereby obtaining the Schwarzschild solution in the limit $b \to 0$.

5. Geodesics, lensing and perihelion precession

The equations of geodesics for the general static spherically symmetric metric is given by [23,24]

$$\dot{r}^2 \equiv \left(\frac{dr}{d\tau}\right)^2 = e^{-\lambda(r)} \left[e^{-\nu(r)} E^2 - \frac{J^2}{r^2} - L \right], \quad (62)$$

$$\dot{\phi} \equiv \frac{d\phi}{d\tau} = \frac{J}{r^2}, \quad (63)$$

$$\dot{t} \equiv \frac{dt}{d\tau} = Ee^{-\nu(r)}, \quad (64)$$

where the motion is considered to be confined in the $\theta = \pi/2$ plane and the constants E and J respectively are energy per unit mass and angular momentum about an axis perpendicular to the invariant plane ($\theta = \pi/2$). Here, τ is an affine parameter and L is the Lagrangian having the values 0 and 1 respectively for null and time-like particles. We do not consider the space-like particles in the present analysis. Now, consider the case $c_1 = 0$.

The metric coefficients e^{ν} and e^{λ} in this case yield the equations for the radial geodesics ($J = 0$)

$$\left(\frac{dr}{dt}\right)^2 = (1 - L/E^2)\left(1 - \frac{b}{r^2}\right), \; \frac{dt}{d\tau} = E, \quad (65)$$

with solution

$$t = \pm \frac{\sqrt{r^2 - b}}{\sqrt{1 - L/E^2}} + constant \quad (66)$$

and the affine parameter $\tau \propto t$. The above equation represents a hyperbola and shows that to an external observer a radially in-falling particle (time-like or null) approaches the radius $r = \sqrt{b}$ asymptotically but can never reach it. As τ is linear in t we find that the $\tau - r$ relationship also represents a hyperbola. Now, for time-like geodesics ($L = 1$), τ is the proper time and hence an observer falling with a time-like particle also avoids the physical singularity at $r = 0$ by asymptotically grazing the critical radius at $r = \sqrt{b}$. This feature is the characteristic of a wormhole space-time and is in sharp contrast with what happens in a Schwarzschild space-time. The equation of orbit in this case is,

$$\left(\frac{du}{d\phi^2}\right)^2 = \left(\frac{E^2 - L}{J^2} - u^2\right)(1 - bu^2), \quad (67)$$

where $u = 1/r$. Now, in order to have bound orbits ($E^2 < 1$) the equation $du/d\phi = 0$ must have at least two real, positive roots which do not coincide with the physical or coordinate singularity. These roots are known to indicate the two turning points of the closed orbit. In the present case it can easily be seen that the real positive values of u for which $du/d\phi$ vanishes are $1/\sqrt{b}$ and E/J for null geodesics ($L = 0$). However, as the metric diverges at

$u = 1/r = 1/\sqrt{b}$ we infer that the null geodesics cannot follow closed orbits in a wormhole space-time However for time-like geodesics ($L = 1$) no positive real value of u exists for which the metric is non-singular and $du/d\phi = 0$ Therefore, bound orbits are not permissible for time-like geodesics also, i e , for all kinds of particles we can only have unbound orbits ($E^2 > 1$) We further observe that,

$$\frac{d^2u}{d\phi^2}+\left[1+\frac{b}{D^2}\right]u = 2bu^3, \quad D = \frac{J}{\sqrt{E^2-L}}, \tag{68}$$

shows that the KR field not only alters the intercept on the ϕ-axis but also produces a departure from the straight line motion due to the term bu^3 on the right of the above equation Although the impact parameter D are different for massive and massless particles, the amount of bending near the origin of the force is same for both kinds of particles. At $r = r_0$, the distance of closest approach towards the origin of force $du/d\phi = 0$, which yields $r_0 = D$. Replacing back u by $1/r$ we obtain from the above equation,

$$\phi(r)-\phi_\infty = \sin^{-1}(r_0/r)+\frac{b}{4r_0^2}\left[\sin^{-1}(r_0/r) - (r_0/r)\sqrt{1-(r_0/r)^2}\right]+O\left(\frac{b}{r_0^2}\right)^2 \tag{69}$$

The bending angle for all types of particles is thus given by

$$\Delta\phi = 2|\phi(r_0)-\phi_\infty|-\pi = \frac{b}{r_0^2}\frac{\pi}{4}+O\left(\frac{b^2}{r_0^4}\right) \tag{70}$$

An exact integration also gives the following expression for the amount of bending

$$\Delta\phi = 2\left(K\left[\frac{b}{r_0^4}\right]\right)-\pi, \tag{71}$$

where $K[x]$ is the complete elliptic integral of the first kind A plot of $\Delta\phi$ as a function of x shows a linear region for small values of $b \ll r_0^2$ with a slope reasonably close to $\pi/4$ – a fact which is demonstrated in the approximate calculation of the bending angle discussed above

In order to obtain information about the trajectories of photons in the geometrical optics limit we can, alternatively, solve for $u(\phi)$, in terms of elliptic functions by directly integrating the equation given below

$$\left[\frac{du}{d\phi}\right]^2 = b\left(\frac{1}{b}-u^2\right)\left(\frac{1}{r_0^2}-u^2\right) \tag{72}$$

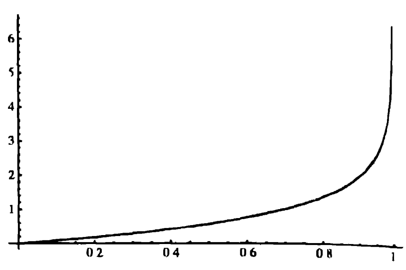

Figure 1 Plot of $\Delta\phi$ *vs* x

For $r_0^2 = b$ the integration yields

$$r(\phi) = \frac{1}{u(\phi)} = \sqrt{b}\coth(\phi-\phi_0). \tag{73}$$

In terms of the proper radial distance $l = \pm\sqrt{r^2-b}$ we have

$$l = \pm\sqrt{b}\,\mathrm{cosech}(\phi-\phi_0). \tag{74}$$

The general solution for $u(\phi)$ is given as

$$(\phi-\phi_0) = \frac{\sqrt{b}}{r_0}F\left[\arcsin r_0 u, \frac{b}{r_0^2}\right], \tag{75}$$

where F now denotes the incomplete elliptic integral of the first kind Inverting this we find

$$u(\phi) = \frac{1}{r^0}sn\left[\frac{r_0}{\sqrt{b}}(\phi-\phi_0), \frac{b}{r_0^2}\right], \tag{76}$$

where sn denotes the Jacobian elliptic function A plot of $r_0 u$ *versus* ϕ with $\phi_0 = 0$ and $r_0/\sqrt{b} = 3$ is shown below

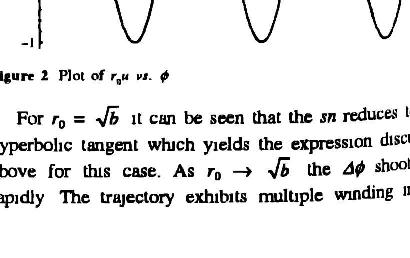

Figure 2 Plot of $r_0 u$ *vs.* ϕ

For $r_0 = \sqrt{b}$ it can be seen that the sn reduces to the hyperbolic tangent which yields the expression discussed above for this case. As $r_0 \to \sqrt{b}$ the $\Delta\phi$ shoots up rapidly The trajectory exhibits multiple winding in the vicinity of the throat. One may conjecture that the negative energy density present in the vicinity of the throat is responsible for such peculiar behaviour It is also clear that for other values of the ratio $x = b/r_0^2$ ($x < 1$) we find that the Kalb-Ramond field deflects the light ray by an amount which is crucially dependent on the KR parameter $\sqrt{b}$

We now consider the case $c_1 \neq 0$ Here we have the complete series solution for the equations with c_1 identified as $-r_s$, the Schwarzschild radius. To study the motion of geodesics in outside r_s we consider the torsion to be small, *i e*, $b/r^2 << 1$ This is based on the assumption that the small torsion (or, equivalently the small KR field energy density) does not completely change the nature of the trajectory of particles as in the case $c_1 = 0$ It rather inflicts a correction over the general relativistic phenomena like bending of light and the perihelic precession of planetary orbits Dropping terms of order cubic or more in r_s/r and b/r^2 we can approximately write the solutions as,

$$e^{\nu} = 1 - \frac{r_s}{r}, \tag{77}$$

$$e^{\lambda} = 1 - \frac{r_s}{r} - \frac{b}{r^2} \tag{78}$$

A similar analysis for the case $c_1 = 0$ shows that for time-like particles closed orbits are possible for the truncated series form of the metric coefficients as discussed above

1 Bending of light rays

For photons the trajectory equations yield

$$\left(\frac{dr}{d\phi}\right)^2 = r^4 e^{-\lambda(r)}\left(\frac{e^{-\nu(r)}}{D^2} - \frac{1}{r^2}\right), \quad D = \frac{J}{E} \tag{79}$$

with solution in the form of a quadrature

$$\phi(r) - \phi_\infty = \int_r^\infty \frac{e^{\lambda(r)/2} dr/r}{\sqrt{e^{-\nu(r)}\frac{r^2}{D^2} - 1}} \tag{80}$$

At the distance of closest approach (r_0) to the center of force, $\left.\frac{dr}{d\phi}\right|_{r=r_0} = 0$, which gives $D^2 = r_0^2 e^{-\nu(r_0)}$. Using this and the specific expressions for the metric components given earlier, we obtain

$$\phi(r) = \phi_\infty + \sin^{-1}(r_0/r) + \frac{r_s}{2r_0}\left(2 - \sqrt{1-(r_0/r)^2} - \sqrt{\frac{r-r_0}{r+r_0}}\right)$$

$$+ \frac{3r_s^2}{8r_0^2}\left[\frac{1}{2}\sin^{-1}\left(\frac{r_0}{r}\right) - 2\cos^{-1}\left(\frac{r_0}{r}\right) - \frac{r_0}{r}\sqrt{1-\left(\frac{r_0}{r}\right)^2} + \frac{3}{2}\sqrt{\frac{r-r_0}{r+r_0}} - \frac{1}{6}\left(\frac{r-r_0}{r+r_0}\right)^{3/2}\right]$$

$$+ \frac{b}{4r_0^2}\left[\sin^{-1}(r_0/r) - (r_0/r)\sqrt{1-(r_0/r)^2}\right]$$

$$+ \textit{higher order terms} \tag{81}$$

The angle of bending is given by,

$$\Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi = \frac{2r_s}{r_0} + \frac{3\pi}{16}\left(\frac{r_s}{r_0}\right)^2 + \frac{b}{r_0^2}\frac{\pi}{4} \tag{82}$$

Looking at the above expression we note that the first and third terms are relevant as long as we are interested in results valid upto first order in the mass M ($r_s \sim GM$) and the KR parameter b. The first term is of course the usual Schwarzschild bending, whereas the third term comes from the KR field Total bending is therefore

$$\Delta\phi = (\Delta\phi)_{\text{Schw}}\left[1 + \frac{(\Delta\phi)_{\text{KR}}}{(\Delta\phi)_{\text{Schw}}}\right] \tag{83}$$

Now the maximum KR energy density is obtained by using the minimum value of r (which is r_0, the impact parameter) This energy density can be written as

$$\left|\rho_{\text{KR}}^{\max}\right| = \frac{c^4}{8\pi G}\frac{b}{r_0^4} = \frac{8}{3\pi}\frac{Mc^2}{V_0}\frac{(\Delta\phi)_{\text{KR}}}{(\Delta\phi)_{\text{Schw}}}, \tag{84}$$

where V_0 is the volume $\frac{4}{3}\pi r_0^3$

We have calculated this energy density using the error bars for the current light bending measurements for Sun [25] The estimated amount of energy per unit volume turns out to be enormous resulting in an unacceptable ambient KR temperature much larger than that for CMBR This suggests that $(\Delta\phi)_{\text{KR}}$ has to be far less than the value of these error bars in order to give a reasonable KR field energy density and therefore remains undetectable within the present day experimental precision.

2. Perihelion precession of planetary orbits

As has been mentioned earlier bound orbits are really possible for time-like particles when $c_1 = 0$. In the case of elliptic planetary orbits, we obtain as in the previous case,

$$\left(\frac{dr}{d\phi}\right)^2 = r^4 e^{-\lambda(r)}\left(\frac{e^{-\nu(r)}E^2 - 1}{J^2} - \frac{1}{r^2}\right) \tag{85}$$

At perihelia and aphelia, $r = r_\mp$ and $\left.\frac{dr}{d\phi}\right|_{r=r_\pm} = 0$, whence the above equation yields

$$\frac{1}{r_+^2} - \frac{e^{\nu(r_+)}E^2}{J^2} = -\frac{1}{J^2} \tag{86}$$

Solving for E and J we obtain the trajectory equation in the form of the quadrature [24]

$$\phi(r) - \phi(r_-) = \int_{r_-}^{r} e^{\lambda(r)/2}(r)\frac{dr}{r^2} \tag{87}$$

where

$$Y(r) = \frac{e^{\nu(r_+)}r^2\left[e^{\nu(r)} - e^{\nu(r_-)}\right] - e^{\nu(r_-)}r_+^2\left[e^{\nu(r)} - e^{\nu(r_+)}\right]}{r_+^2 r^2 e^{\nu(r)}\left[e^{\nu(r_+)} - e^{\nu(r_-)}\right]} - \frac{1}{r^2}. \tag{88}$$

From these the amount of rotation of the orbit per revolution becomes,

$$\Delta\phi = 2|\phi(r_+) - \phi(r_-)| - 2\pi$$

$$= \left[\frac{3r_s}{l} + \frac{3r_s^2}{8l^2}(18 + e^2) + \frac{b}{l^2}\left(1 + \frac{e^2}{2}\right)\right]\pi$$

$$+ \textit{higher order corrections}. \tag{89}$$

Here L and e are the semilatus rectum and eccentricity of the elliptic orbit which are defined as $2/l = (1/r_+ + 1/r_-)$, $r_\pm = (1 \pm e)a$, a being the semi-major axis

Following the similar path as in the preceding section the maximum KR energy density obtained by using the minimum value of r (which in this case is the perihelion distance r_-) is given by

$$\left|\rho_{\mathrm{KR}}^{\max}\right| = \frac{c^4}{8\pi G}\frac{b}{r^4} \sim \frac{Mc^2}{V}\frac{(\Delta\phi)_{\mathrm{KR}}}{(\Delta\phi)_{\mathrm{Schw}}} \tag{90}$$

where we have considered the eccentricity e to be fairly small so that $r \sim l$, the semilatus rectum, and V is the volume $\frac{4}{3}\pi l^3$ of the two-body system Comparing with the standard observational data for the perihelic precession of Mercury, Earth and Icarus [25] we have calculated $\left|\rho_{\mathrm{KR}}^{\max}\right|$ with $(\Delta\phi)_{\mathrm{KR}}$ of the order of error bars Once again the energy density is found to be extremely high, thus suggesting that $(\Delta\phi)_{\mathrm{KR}}$ should be much less than the present day experimental error bars

6. Conclusions

In this paper we have described how the presence of space-time torsion modifies the predictions pure Einsteinian gravity in the context of astrophysical/cosmological observations One of the major obstacle in considering the effects of torsion on electromagnetic phenomena in an Einstein-Cartan framework was the loss of $U(1)$ gauge invariance We have explained how a string inspired supergravity-based model can help to overcome the problem from the requirement of quantum consistency of such theories We have then described how the hitherto unexplained wavelength independent rotation of the plane of polarization of distant galactic electromagnetic radiation can be accounted for in a space-time with torsion In addition we explicitly work out the most general spherically symmetric asymptotically flat solution for the space-time metric when torsion is present This generalizes the well-known Schwarzschild solution which is the unique vacuum solution of the Einstein's equation in absence of torsion This naturally motivates us to recalculate the predicted experimental measure of the well-known tests of Einsteinian gravity like perihelion precession of planetary orbits and bending of light change in presence of torsion Using the experimentally measured value we can put an upper bound on the energy density of the source of the torsion namely the KR field The bound turns out to be much to small and certainly much beyond the scope of the present day experiments However this work clearly points out the departure from the Einsteinian framework of gravity in the realm of the well-known cosmological/astrophysical observables when torsion is present in the space-time various other aspects of the presence of torsion in the context of gravitational redshift, late time acceleration of the Universe, cosmic microwave background anisotropy has been studied extensively both in four and higher dimensional models of gravity [26] With the development of our experimental accuracy we hope to arrive at a situation when we shall be able pin down whether our space-time is purely Riemannian or not

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